Lecture 7 **2012/11/12**

Outline

1. **Motivation Example**
2. **Basic Properties of** e^A
3. **Role of** e^{At} **and Computation**
4. **Summary**

1. Motivation Example

For the linear system as follows

$$x' = A(t)x + h(t), \quad x(t_0) = x_0,$$

we have an algebraic structure of solution given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s,$$

where $\Phi(t)$ is a fundamental matrix solution. However, this result doesn't tell us how to find $\Phi(t)$. In fact, it has no way to get an explicit solution in general. See the example of Ricatti equation

$$x' = t^2 + x^2,$$

which has no explicit solution. Taking a transformation: $x = -\frac{u'}{u}$, we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

This is a type of form: $x' = A(t)x$. We conclude that we are not able to solve the above particular linear time-varying system. Otherwise, we can solve the Ricatti equation. It shows that we can't expect to solve $\Phi(t)$ for $x' = A(t)x$ in general without restriction. There is no way to find a systematic method to solve $\Phi(t)$.

If $A(t) \equiv A$, we are able to find its fundamental matrix solution $\Phi(t) = e^{At}$, which is an exponential of the matrix A.

The following questions need to solve:

- Definition of e^A and Basic Properties;

- Role of e^{At} and its Computation;

2. Basic Properties of e^{A}

Definition 7.1 For a real $n\times n$ matrix A, define

$$e^{A}=I_n+A+\frac{A^2}{2!}+\cdots+\frac{A^m}{m!}+\cdots\quad=\sum_{k=0}^{\infty}\frac{A^k}{k!},$$

where $A^0=I_n$.

Remark 7.1 Since $\|I_n\|=1$, $\|A^m\|\le\|A\|^m$, the series

$$1+\|A\|+\frac{\|A\|^2}{2!}+\cdots+\frac{\|A\|^m}{m!}+\cdots=e^{\|A\|}$$

is convergent. Therefore, e^{A} is well defined.

Remark 7.2 Unlike the exponential of scalar e^{a}, e^{A} will be much complicated.

Proposition 7.1 Let A, B, P be all $n\times n$ matrices. Then

1) If $AB=BA$, then $e^{A+B}=e^{A}e^{B}$.

2) For any $A\in L(K^n)$, e^{A} is invertible and $(e^{A})^{-1}=e^{-A}$.

3) If P is invertible, then $e^{P^{-1}AP}=P^{-1}e^{A}P$.

Proof. 1) Since $AB=BA$, the binomial theorem holds for matrix, that is,

$$(A+B)^k=\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}A^jB^{k-j}.$$

Then,

$$e^{A+B}=\sum_{k=0}^{\infty}\frac{(A+B)^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\{\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}A^jB^{k-j}\}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{A^jB^{k-j}}{j!(k-j)!}.$$

Meanwhile,

$$e^{A}=\sum_{k=0}^{\infty}\frac{A^k}{k!}\ ;\quad e^{B}=\sum_{j=0}^{\infty}\frac{B^j}{j!},$$

we have

$$e^A e^B = (\sum_{k=0}^{\infty} \frac{A^k}{k!}) \ (\sum_{j=0}^{\infty} \frac{B^j}{j!}) \overset{\Delta}{=} \sum_{k=0}^{\infty} c_k \ ,$$

where c_k is a Cauchy product as follows.

$$c_k = \frac{A^0}{0!} \frac{B^k}{k!} + \frac{A^1}{1!} \frac{B^{k-1}}{(k-1)!} + \cdots + \frac{A^{k-1}}{(k-1)!} \frac{B^1}{1!} + \frac{A^k}{k!} \frac{B^0}{0!} = \sum_{j=0}^{k} \frac{A^j B^{k-j}}{j!(k-j)!}$$

Therefore, $e^{A+B} = e^A e^B$.

2) Since A and $-A$ are always commutative, putting $B = -A$ in 1) gives

$$I_n = e^0 = e^{A+(-A)} = e^A e^{-A},$$

which implies that e^A is invertible and $(e^A)^{-1} = e^{-A}$.

3) If P is invertible, then

$$e^{PAP^{-1}} = \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!} = I + \sum_{k=1}^{\infty} \frac{(PAP^{-1})^k}{k!}$$

$$= I_n + P(\sum_{k=1}^{\infty} \frac{A^k}{k!}) P^{-1} = P(\sum_{k=0}^{\infty} \frac{A^k}{k!}) P^{-1} = P^{-1} e^A P. \quad \square$$

Remark 7.3 e^A enjoys the property of e^a when $AB = BA$. e^A invertible is analogous to the fact that $e^a \neq 0$.

3. Role of e^{At} and its Computation

By the definition of e^A, we have

$$e^{At} = I_n + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^m t^m}{m!} + \cdots, \ \ t \in R,$$

which is uniformly convergent on any $[-h, h] \subset R$, $h > 0$.

Theorem 7.1 (Fundamental Theorem of Linear System with Constant Coefficient) e^{At} is a principle matrix solution of $x' = Ax$.

Proof. Denote $\Phi(t) = e^{At}$. Obviously $\Phi(0) = I_n$. Since e^{At} is uniformly convergent on any compact interval of R, we have

$$\Phi'(t) = (e^{At})' = A + A^2 t + \frac{A^3 t^2}{2!} + \cdots + \frac{A^m t^{m-1}}{(m-1)!} + \cdots$$

$$= A(I_n + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^m t^m}{m!} + \cdots)$$

$$= Ae^{At} = A\Phi(t).$$

Therefore, e^{At} is a principle matrix solution of $x' = Ax$. □

Remark 7.4 Although e^{At} is well-defined and is a principle matrix solution of $x' = Ax$, its computation is still nontrivial because e^{At} is the form of infinite series.

Example 7.1 If $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $J^2 = -I_2$ and thus by induction $J^{2n} = (-1)^n I_2$ and $J^{2n+1} = (-1)^n J$. We have

$$e^{tJ} = \sum_{j=0}^{\infty} \frac{t^j J^j}{j!} = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} J^{2j} + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} J^{2j+1}$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} (-1)^j I_2 + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} (-1)^j J$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} (-1)^j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} (-1)^j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Example 7.2 If $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, $b \neq 0$, then we have

$$e^{At} = e^{bJt} = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}.$$

Example 7.3 If $A=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $b \neq 0$, then $A = aI_2 + bJ$ and we have

$$e^{At} = e^{aI_2 t + bJt} = e^{aI_2 t} e^{bJt} = e^{at}\begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}.$$

Example 7.4 If $A=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, then $A = 2I_2 + Z$, where $Z=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with $Z^2 = O$.

Since I_2 and Z commute, then

$$e^{At} = e^{(2I_2+Z)t} = e^{2I_2 t} e^{Zt} = e^{2t} e^{Zt},$$

where

$$e^{Zt} = I_2 + Zt + \frac{Z^2 t^2}{2!} + \cdots + \frac{Z^m t^m}{m!} + \cdots = I_2 + Zt = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$e^{At} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Remark 7.5 Notice that e^{Zt} is a finite form in Example 7.4, which can be computed. In fact, Z is nilpotent of order 2, i.e. $Z^2 = O$.

Definition 7.2 A $n \times n$ matrix N is said to be **nilpotent** of order k if $N^k = O$ and $N^{k-1} \neq O$.

Definition 7.3 Let λ be an eigenvalue of A. We define

1) λ has an **algebraic multiplicity** of l if λ is a zero of order l of $P(\lambda) = \det(A - \lambda I)$;
2) λ has a **geometric multiplicity** of k if k is the dimension of the subspace spanned by the eigenvectors of A for λ, i.e. the number of the existed linearly independent eigenvectors belongs to λ, denoted by $k = \dim \operatorname{Ker}(A - \lambda I_n)$, where $\operatorname{Ker}(A - \lambda I_n) \overset{def.}{=} \{v \in R^n \mid (A - \lambda I_n)v = 0\}$ is **the kernel** of $A - \lambda I_n$.

Remark 7.6 Clearly $k \le l$. If $k = l$, A is diagonal.

Definition 7.4 Let λ be an eigenvalue of A. The **generalized eigenspace** of λ consists of the subspace

$$E_\lambda = \{v \in R^n : (A - \lambda I_n)^k v = 0, \text{ some } k \in N^+\}.$$

The elements of the generalized eigenspace are called **generalized eigenvectors**.

Lemma 7.1 E_λ is invariant under A.

Proof. We need to show that $\forall\, v \in E_\lambda \Rightarrow Av \in E_\lambda$. If $v \in E_\lambda$, we have $(A - \lambda I_n)^k v = 0$. Then, $(A - \lambda I_n)^k Av = (A - \lambda I_n)^k Av - \lambda (A - \lambda I_n)^k v = (A - \lambda I_n)^{k+1} v = 0$, and thus $Av \in E_\lambda$. □

Proposition 7.2 Let A be a $n \times n$ matrix. Then there exists a basis of C^n, which consists of generalized eigenvectors, i.e.

$$C^n = \bigoplus_\lambda E_\lambda.$$

Remark 7.7 If A is a real matrix, then there exists a basis of R^n, which consists of generalized eigenvectors, i.e.

$$R^n = \bigoplus_\lambda E_\lambda,$$

where λ may be real or complex.

Definition 7.5 The matrix A is said **semi-simple** or **diagonalizable** if for each λ, algebraic and geometric multiplicity coincide, i.e. $l = k$ for each λ.

Theorem 7.2 (Decomposition Theorem) Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ repeated according to their (algebraic) multiplicity. Then, there exists the decomposition

$$A = S + N,$$

where the matrix S is semi-simple, the matrix N is nilpotent of order k no less than the maximum of the algebraic multiplicities of the eigenvalues, and commute with S, i.e. $SN = NS$.

Proof. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of generalized eigenvectors for C^n by Proposition 7.2. Let $P = (v_1, v_2, \cdots, v_n)$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_j = \lambda$

if $v_j \in E_\lambda$ and P is invertible. Then, we define

$$S = P\Lambda P^{-1} \text{ and } N = A - S .$$

This provides the decomposition $A = S + N$. By construction, S is semi-simple.

Next we show that $SN = NS$. Since $SN - NS = S(A-S)-(A-S)S = SA - AS$,

It suffices to show that $SA = AS$. If $v \in E_\lambda$, then $Av \in E_\lambda$ and $Sv = Av = \lambda v$.

$$(SA - AS)v = SAv - A\lambda v = (S - \lambda I_n)Av = 0 .$$

For $\forall\, v \in C^n$, v is a sum (linear combination) of generalized eigenvectors, we have

$$(SA - AS)v = 0$$

for any $v \in C^n$. So we obtain $SA - AS = O$.

Finally we show that N is nilpotent. Choose k to be larger than or equal to the largest algebraic multiplicity of the eigenvalues of A. If $v \in E_\lambda$, we have $Sv = \lambda v$.

$$N^k v = (A-S)^k v = (A-S)^{k-1}(A - \lambda I_n)v$$

$$= (A-\lambda I_n)(A-S)^{k-1} v = \cdots = (A - \lambda I_n)^k v = 0 .$$

It is the same to get from $N^k v = 0$ for $v \in E_\lambda$ to $N^k v = 0$ for any $v \in C^n$. So $N^k = O$. □

If A is a real matrix with repeated real eigenvalues, we have the following form of Decomposition Theorem.

Theorem 7.3 (Decomposition Theorem) Let A be a **real** $n \times n$ matrix with **real** eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ repeated according to their (algebraic) multiplicity. Then, there exists the decomposition

$$A = S + N ,$$

where the matrix S is semi-simple, i.e. $S = P diag(\lambda_j) P^{-1}$; N is nilpotent of order k no less than the maximum of the algebraic multiplicities of the eigenvalues, and commute with S, i.e. $SN = NS$.

Corollary 7.1 Based on Theorem 7.3, $x' = Ax$ with $x(0) = x_0$ has the solution

$$x(t) = Pdiag(e^{\lambda_j t})P^{-1}[I_n + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}]x_0 .$$

Proof. According to Decomposition Theorem, the fundamental theorem and 3) in Proposition 7.1, we have

$$x(t) = e^{At}x_0 = e^{(S+N)t}x_0 = e^{St}e^{Nt}x_0 = e^{Pdiag\{\lambda_j\}P^{-1}t}e^{Nt}x_0$$

$$= Pdiag(e^{\lambda_j t})P^{-1}[I_n + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}]x_0 . \quad \square$$

Corollary 7.2 If λ has multiplicity n of A, then $x' = Ax$ with $x(0) = x_0$ has the solution

$$x(t) = e^{\lambda t}[I_n + Nt + \cdots + \frac{N^{n-1}t^{n-1}}{(n-1)!}]x_0 .$$

Proof. Since λ has multiplicity n of A, Corollary 7.1 gives

$$x(t) = Pdiag(e^{\lambda t})P^{-1}[I_n + Nt + \cdots + \frac{N^{n-1}t^{n-1}}{(n-1)!}]x_0$$

$$= Pe^{\lambda t}I_n P^{-1}[I_n + Nt + \cdots + \frac{N^{n-1}t^{n-1}}{(n-1)!}]x_0$$

$$= e^{\lambda t}[I_n + Nt + \cdots + \frac{N^{n-1}t^{n-1}}{(n-1)!}]x_0 .$$

Since P could be any basis of R^n here, we take P as I_n, the usual basis for R^n. Then $S = diag(\lambda)$ and $N = A - S$. $\square$

Remark 7.8 In case λ has multiplicity n of A, the solution is particularly easy to be computed without finding a required basis. .

Example 7.5 Solve $x' = Ax$ with $x(0) = x_0$, where $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution. A has $\lambda_1 = \lambda_2 = 2$. Thus, $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $N = A - S = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Since $N^2 = O$, therefore,

$$x(t) = e^{At}x_0 = e^{2t}\ (I_2 + Nt)\,x_0 = e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} x_0 .$$

Example 7.6 Solve $x' = Ax$ with $x(0) = x_0$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Solution. A has $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$. For $\lambda_1 = 1$, we find an eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$; For $\lambda = 2$, we only can find an eigenvector $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We need to find another generalized eigenvector for $\lambda = 2$, independent of v_2 by solving

$$(A - 2\lambda)^2 v = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} v = 0,$$

which yields $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Then,

$$P = \begin{pmatrix} v_1, & v_2, & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Then we compute

$$S = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}; \quad N = A - S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},$$

and $N^2 = O$. The solution is then given by

$$x(t) = P\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} P^{-1}[I_3 + Nt]\,x_0$$

$$= \begin{pmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{pmatrix} x_0 .$$

If A is a real matrix with repeated complex eigenvalues, we have the following form of Decomposition Theorem.

Theorem 7.4 (Decomposition Theorem) Let A be a **real** $2m\times 2m$ ($2m=n$) matrix with **complex** eigenvalues $\lambda_j=\alpha_j+i\beta_j$ and $\bar{\lambda}_j=\alpha_j-i\beta_j$, $j=1,2,\cdots,m$, repeated according to their (algebraic) multiplicity. Then, there exists a basis of generalized complex eigenvectors $w_j=u_j+iv_j$ and $\bar{w}_j=u_j-iv_j$, $j=1,2,\cdots,m$ for C^n and $\{u_1,v_1,\cdots,u_m,v_m\}$ is a basis for R^n. For any such a basis, $P=(u_1,v_1,\cdots,u_m,v_m)$ is invertible and the decomposition

$$A=S+N,$$

where the matrix S is diagonal blocks, i.e. $S=Pdiag\left(\begin{bmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{bmatrix}\right)P^{-1}$; N is nilpotent of order k no less than the maximum of the algebraic multiplicities of the eigenvalues, and commute with S, i.e. $SN=NS$.

Corollary 7.2 Based on Theorem 7.4, $x'=Ax$ with $x(0)=x_0$ has the solution

$$x(t)=P\,diag(e^{\alpha_j t}\begin{bmatrix}\cos\beta_j t & \sin\beta_j t\\ -\sin\beta_j t & \cos\beta_j t\end{bmatrix})P^{-1}[I_n+Nt+\cdots+\frac{N^{k-1}t^{k-1}}{(k-1)!}]x_0.$$

Proof. According to Decomposition Theorem, the fundamental theorem and 3) in Proposition 7.1, we have

$$x(t)=e^{At}x_0=e^{(S+N)t}x_0=e^{St}e^{Nt}x_0=\exp\{P\,diag\left(\begin{bmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{bmatrix}\right)P^{-1}t\}e^{Nt}x_0$$

$$=P\,diag(e^{\alpha_j t}\begin{bmatrix}\cos\beta_j t & \sin\beta_j t\\ -\sin\beta_j t & \cos\beta_j t\end{bmatrix})P^{-1}[I_n+Nt+\cdots+\frac{N^{k-1}t^{k-1}}{(k-1)!}]x_0. \quad \square$$

Example 7.7 Solve $x'=Ax$ with $x(0)=x_0$, where $A=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 2 & 0 & 1 & 0\end{pmatrix}$.

Solution. A has $\lambda=i$ and $\bar{\lambda}=-i$ of multiplicity 2. The equation

$$(A-\lambda I_4)w=\begin{pmatrix}-i & -1 & 0 & 0\\ 1 & -i & 0 & 0\\ 0 & 0 & -i & -1\\ 2 & 0 & 1 & -i\end{pmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}=0$$

is equivalent to $z_1=z_2=0$ and $z_3=iz_4$. Thus, we have one eigenvector $w_1=\begin{pmatrix}0, & 0, & i, & 1\end{pmatrix}^T$. Also the equation

$$(A-\lambda I_4)^2 w=\begin{pmatrix}-2 & 2i & 0 & 0\\ -2i & -2 & 0 & 0\\ -2 & 0 & -2 & 2i\\ -4i & -2 & -2i & -2\end{pmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}=0$$

is equivalent to $z_1=iz_2$ and $z_3=iz_4-z_1$. We therefore choose the generalized eigenvector $w_2=\begin{pmatrix}i, & 1, & 0, & 1\end{pmatrix}^T$. Taking real and imaginary part of w_1 and w_2 gives

$$u_1=\begin{pmatrix}0, & 0, & 0, & 1\end{pmatrix}^T;\ v_1=\begin{pmatrix}0, & 0, & 1, & 0\end{pmatrix}^T;$$

$$u_2=\begin{pmatrix}0, & 1, & 0, & 1\end{pmatrix}^T;\ v_2=\begin{pmatrix}1, & 0, & 0, & 0\end{pmatrix}^T.$$

According to Decomposition Theorem, we have

$$P=(u_1,v_1,u_2,v_2)=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\end{pmatrix};\quad P^{-1}=\begin{pmatrix}0 & -1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix};$$

$$S=Pdiag\left(\begin{bmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{bmatrix}\right)P^{-1}$$

$$=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}\begin{pmatrix}0 & -1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 1 & 0 & 1 & 0\end{pmatrix};$$

$$N=A-S=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\ \text{with}\ N^2=O.$$

Thus, the solution is given by

$$x(t) = P\begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} P^{-1}[I_4 + Nt]x_0$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} x_0$$

$$= \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ -t\sin t & \sin t - t\cos t & \cos t & -\sin t \\ \sin t + t\cos t & -t\sin t & \sin t & \cos t \end{pmatrix} x_0 .$$

Remark 7.9 If A has both real and complex repeated eigenvalues, a combination of Theorem 7.3 and Theorem 7.4 can be used. See the following example for how.

Example 7.7 Solve $x' = Ax$ with $x(0) = x_0$, where $A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{pmatrix}$.

Solution. A has $\lambda_1 = -3$, $\lambda_2 = 2+i$ with $\bar{\lambda}_2 = 2-i$. The corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad w_2 = u_2 + i v_2 = \begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}; \quad N = O, \quad S = A.$$

The solution is given by

$$x(t) = P\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\alpha_2 t}\cos\beta_2 t & e^{\alpha_2 t}\sin\beta_2 t \\ 0 & -e^{\alpha_2 t}\sin\beta_2 t & e^{\alpha_2 t}\cos\beta_2 t \end{pmatrix} P^{-1}x_0$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}\cos t & e^{2t}\sin t \\ 0 & -e^{2t}\sin t & e^{2t}\cos t \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} x_0$$

$$= \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\cos t + \sin t) & -2e^{2t}\sin t \\ 0 & e^{2t}\sin t & e^{2t}(\cos t - \sin t) \end{pmatrix} x_0 .$$

Remark 7.10 There are several ways to compute e^{At}, which is a finite form in fact. The decomposition method gives a clear algebra structure property. P is a basis of generalized eigenvectors, S is semi-simple (diagonalizable) and $A = S + N$, where N is nilpotent, $SN = NS$. Although the Jordan form method, the Putzer algorithm and the others can work for computing e^{At}, which are not listed here. They don't have such a nice structure decomposition property.

4. Summary

- e^{At} plays a key role in linear systems with constant coefficient. Its computation is completely solved by Decomposition Theorems.

- For solving $x' = Ax + h(t)$, $x(0) = x_0$, we have the formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} h(s)ds .$$

Homework

Problem 7.1 The "Putzer Algorithm" given below is another method for computing e^{At} when we have multiple eigenvalues:

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t)P_j \text{ ,}$$

where $P_0 = I_n$, $P_j = (A - \lambda_j I_n)(A - \lambda_{j-1} I_n) \cdots (A - \lambda_1 I_n)$, $j = 1, 2, \cdots, n$, and $r_j(t)$, $j = 1, 2, \cdots, n$, are the solutions of the first-order linear differential equations and initial conditions

$$r_1'(t) = \lambda_1 r_1(t) \quad \text{with} \quad r_1(0) = 1 ;$$

$$r_2'(t) = \lambda_2 r_2(t) + r_1(t) \quad \text{with} \quad r_2(0) = 0 ;$$

$$\cdots ;$$

$$r_n'(t) = \lambda_n r_n(t) + r_{n-1}(t) \quad \text{with} \quad r_n(0) = 0 .$$

1) Use the Putzer Algorithm to compute e^{At} for the matrix A given in Example 7.5 and Example 7.6.
2) Can you show the Putzer Algorithm? You are encouraged to do it. (Selective) (Hint: $P_n = (A - \lambda_n I_n)(A - \lambda_{n-1} I_n) \cdots (A - \lambda_1 I_n) = O_{n\times n}$)

Problem 7.2 If $J = diag(J_j)$, where J_j is a matrix of order $n_j\ (>0)$ and $\sum_{j=1}^{r} n_j = n$, show that $e^{Jt} = diag(e^{J_j t})$.

Problem 7.3 If $J = \begin{pmatrix} \lambda & 1 & 0 & \ddots & 0 \\ 0 & \lambda & 1 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ddots & \ddots & 0 & \lambda \end{pmatrix}_{m\times m}$, show that

$$e^{Jt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 & t \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{m\times m} .$$

Problem 7.4 If $J = diag(J_j)$, where

$$J_j = \begin{pmatrix} \lambda_j & 1 & 0 & \ddots & 0 \\ 0 & \lambda_j & 1 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ddots & \ddots & 0 & \lambda_j \end{pmatrix}_{n_j\times n_j}$$

is a Jordan matrix block of order n_j (>0) with $\sum_{j=1}^{r} n_j = n$, show that $e^{Jt} = diag(e^{J_j t})$, where

$$e^{J_j t} = e^{\lambda_j t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_j - 1}}{(n_j - 1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_j - 2}}{(n_j - 2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 & t \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{n_j \times n_j} .$$